The Scattering Amplitude for the Shrödinger Operator in a Layer

Michel Cristofol, Patricia Gaitan

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Abstract

For a layer with compactly supported inhomogeneity we give an appropriate definition of the scattering amplitude. Then we recover the relation between the far field pattern and the scattering amplitude.

1 Introduction

The scattering amplitude is a wellknown way to solve some inverse scattering problem. In the litterature we actually find the definition of the scattering amplitude in the whole homogeneous perturbed space [5],[3] or half space [1]. For a stratified space, this definition is more complicated [2]. The situation in the layer, not yet studied, introduces also difficulties. In this paper we are looking for a definition of the scattering amplitude in the layer. Such wave guides can modelize problems of wave propagation in geophysic, underwater acoustic and so on.
One consider the Schrödinger operator $A = -\Delta + V(X)$ in $L^2(\Omega)$ which derives from $A_0 = -\Delta$ in $L^2(\Omega)$, where $\Omega = \mathbb{R}^n \times (0, \pi)$, $n \geq 1$ and $D(A) = D(A_0) = H_0^1(\Omega) \cap H^2(\Omega)$. We study the boundary value problem:

$$\left\{ \begin{array}{l} (-\Delta + V(X))u(X) = \lambda u(X) \quad \text{in} \quad L^2(\Omega), \\ u(X)_{|\partial\Omega} = 0. \end{array} \right. \tag{1.1}$$

We define the scattering amplitude and the outgoing solutions of (1.1). A work in progress uses the scattering amplitude to establish uniqueness theorem for the potential $V(X)$.
The paper is organized as follow. In section 2, we carry out the generalized eigenfunctions of A and we define the wave operators. We state results on the existence and completeness of these operators. This allows us to prove that the system of generalized eigenfunctions of A is dense. In section 3, we define the scattering operator, the scattering matrix and finally the scattering amplitude. In section 4, we give an asymptotic development of the generalized eigenfunctions of A and we link it with the scattering amplitude.

2 Spectral Study of A

2.1 Generalized Eigenfunctions of A_0

By a standard Fourier technique, we obtain the generalized eigenfunctions ψ_j^0 of A_0 associated to the eigenvalues $\lambda_j(|p|)$ in the following form

$$\psi_j^0(x', y, p) = (2\pi)^n \sqrt{\frac{2}{\pi}} e^{ix' \cdot p} sin(jy), \quad \text{and} \quad \lambda_j(|p|) = j^2 + |p|^2, \quad j \geq 1,$$

with $X = (x', y)$, $x' = (x_1, ..., x_n) \in \mathbb{R}^n$, $y \in (0, \pi)$ and $p \in \mathbb{R}^n$.
This family of generalized eigenfunctions $\{\psi_j^0\}_{j\geq 1}$ is dense in $L^2(\Omega)$. Then to obtain a spectral representation

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of A_0, if $f \in L^2(\Omega)$, we define the following limit in $L^2(\Omega)$ and the spaces $\mathcal{H}_j$:

$$\widetilde{f}_j(p) = \lim_{M\to+\infty} \int_{|x'|<M} \overline{\psi_j^0(x',y,p)} f(x',y)dx'dy,$$

$$\mathcal{H}_j = \{u \in L^2(\Omega); \exists \widetilde{f}_j \in L^2(\mathbb{R}^n), u(x',y) = \int_{\mathbb{R}^n} \psi_j^0(x',y,p)\widetilde{f}_j(p)dp\}. \tag{2.1}$$

The unitary map $\Phi\ :\ f \mapsto (\widetilde{f}_1(p), \widetilde{f}_2(p), ...)$ from $L^2(\Omega)$ into $\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus ...$ is a spectral representation of A_0 in the following sense

$$\Phi(A_0 f) = (\widetilde{A_0 f_1}, \widetilde{A_0 f_2}, ...) = (\lambda_1(|p|)\widetilde{f}_1(p), \lambda_2(|p|)\widetilde{f}_2(p), ...).$$

2.2 Generalized Eigenfunctions of A

We look for the generalized eigenfunctions of A in the form:

$$\psi_j^{\pm}(x',y,p) = \psi_j^0(x',y,p) + w_j^{\pm}(x',y,p), \tag{2.2}$$

We denote

$$N(\lambda) = \sharp\{j > 1;\ \jmath^2 \leq \lambda\}. \tag{2.3}$$

The function $\psi_j^{\pm}$ satisfies the eigenvalue equation

$$(-\Delta + V(x',y) - \lambda_j(|p|))\psi_j^{\pm}(x',y,p) = 0. \tag{2.4}$$

Then, taking into account the equations (2.2), (2.4) and the limiting absorption principle, we obtain

$$w_j^{\pm}(x',y,p) = -R^{\pm}(\lambda_j(|p|))(V\psi_j^0)(x',y,p), \tag{2.5}$$

where $R^{\pm}(z)$ is the resolvent of A. If we introduce the resolvent equation in (2.5), we obtain

$$w_j^{\pm}(x',y,p) = -R_0^{\pm}(\lambda_j(|p|))(V\psi_j^0)(x',y,p) + R_0^{\pm}(\lambda_j(|p|))VR^{\pm}(\lambda_j(|p|))(V\psi_j^0)(x',y,p). \tag{2.6}$$

2.3 Wave Operators

We first consider the wave operators

$$W_{\pm} = s - \lim_{t\to\pm\infty} e^{itA}e^{-itA_0}, \tag{2.7}$$

and the scattering operator $S = W_+^* W_-$. The wave operators actually exist. The proof is an adaptation of the theorem XIII-31 of [6]. Then the limits (2.7) exist, moreover we obtain the completeness of the wave operators $W_{\pm}$ from $\mathcal{H}_{ac}^0(\Omega) = L^2(\Omega)$ (subspace of absolute continuity of A_0) into $\mathcal{H}_{ac}(\Omega)$ (subspace of absolute continuity of A), see [4]. To obtain a spectral representation of A, if $f \in L^2(\Omega)$, we define the following limit in $L^2(\Omega)$:

$$\widetilde{f_j^{\pm}}(p) = \lim_{M\to+\infty} \int_{|x'|<M} \overline{\psi_j^{\pm}(x',y,p)} f(x',y)dx'dy,$$

and $\mathcal{H}_j^{\pm} = W_{\pm}\mathcal{H}_j$, where $\mathcal{H}_j$ is defined by (2.1). The unitary map $\Phi^{\pm}\ :\ f \mapsto (\widetilde{f_1^{\pm}}(p), \widetilde{f_2^{\pm}}(p), ...)$ from $L^2(\Omega)$ to $\mathcal{H}_1^{\pm} \oplus \mathcal{H}_2^{\pm} \oplus ...$ is a spectral representation of $A_{|\mathcal{H}_{ac}}$.

3 The Scattering Amplitude in a Layer

We have defined in the previous section the scattering operator $S = W_+^* W_-$. Using the relations $\Phi^+ = \Phi W_-^*$ and $\Phi^- = \Phi W_+^*$, the scattering operator can be written

$$S = \Phi^* \Phi^- \Phi^{+*} \Phi. \tag{3.1}$$

In a first step we define the scattering matrix as follows

$$\widetilde{Su_j}(p) = (\widetilde{S}(\lambda_j(|p|))u_j)(\omega) \quad \text{with} \quad \omega \in S^{n-1}. \tag{3.2}$$

Note that $\widetilde{S}(\lambda) \in \mathcal{L}((L^2(S^{n-1}))^{N(\lambda)})$. We obtain an $N(\lambda) \times N(\lambda)$-matrix, where $N(\lambda)$ was defined by (2.3). So we have to determine $\widetilde{Su_j}(p)$. Using (3.1) we obtain $\Phi S u = \Phi u + (\Phi^- - \Phi^+)\theta^+ u$ where $\theta^+ = \Phi^{+*}\Phi$. Then we have

$$\widetilde{Su_j}(p) = \widetilde{u_j}(p) + \lim_{M \to +\infty} \int_{|x'|<M,\ y \in (0,\pi)} \overline{[\psi_j^-(x', y, p) - \psi_j^+(x', y, p)]}\theta^+ u(x', y) dx' dy.$$

After several technical transformations, we obtain:

$$\begin{aligned} &\widetilde{Su_j}(p) = \widetilde{u_j}(p) + \textstyle\sum_{k^2<j^2+|p|^2} i\pi\alpha^{n-2} \\ &\Big(\int_{\mathbb{R}^n \times (0,\pi)} V(x', y)\psi_j^0(x', y, p) \int_{S^{n-1}} \overline{\psi_k^0(x', y, \alpha\omega')}\widetilde{u_k}(\alpha\omega')d\omega' dx' dy \\ &- \int_{\mathbb{R}^n \times (0,\pi)} V(x', y)\psi_j^0(x', y, p) \int_{S^{n-1}} \overline{[R^+(\lambda_k(\alpha\omega'))V\psi_k^0(\cdot, \cdot, \alpha\omega')]}(x', y, \alpha\omega')\widetilde{u_k}(\alpha\omega')d\omega' dx' dy, \end{aligned} \tag{3.3}$$

with $\alpha = (\lambda_j(|p|) - k^2)^{1/2}$. So taking into account (3.2) and (3.3) we obtain explicitely the scattering matrix and the scattering amplitude $\mathcal{A}(\omega, \omega', \lambda_j(|p|)) = [\mathcal{A}_{j,k}(\omega, \omega', \lambda_j(|p|))]_{1 \le j,k \le N(\lambda)}$ is defined as the kernel of this operator

$$((S(\widetilde{\lambda_j(|p|)}) - I)\widetilde{u_j})(\omega) = \int_{S^{n-1}} \sum_{k^2<j^2+|p|^2} \mathcal{A}_{j,k}(\omega, \omega', \lambda_j(|p|))\widetilde{u_k}(\alpha\omega')d\omega'. \tag{3.4}$$

Proposition 3.1 *The scattering amplitude is a $N(\lambda) \times N(\lambda)$-matrix*

$$\mathcal{A}(\omega, \omega', \lambda) = [\mathcal{A}_{j,k}(\omega, \omega', \lambda_j(|p|))]_{1 \le j,k \le N(\lambda)}$$

$$\mathcal{A}_{j,k}(\omega, \omega', \lambda_j(|p|)) = \mathcal{B}_{j,k}(\omega, \omega', \lambda_j(|p|)) + \mathcal{C}_{j,k}(\omega, \omega', \lambda_j(|p|)), \tag{3.5}$$

$$\mathcal{B}_{j,k} = i\pi\alpha^{n-2} \int_{\mathbb{R}^n \times (0,\pi)} V(x', y)\psi_j^0(x', y, p) \int_{S^{n-1}} \overline{\psi_k^0(x', y, \alpha\omega')}\widetilde{u_k}(\alpha\omega')d\omega' dx' dy$$

$$\mathcal{C}_{j,k} = -i\pi\alpha^{n-2} \int_{\mathbb{R}^n \times (0,\pi)} V(x', y)\psi_j^0(x', y, p) \int_{S^{n-1}} \overline{[R^+(\lambda_k(\alpha\omega'))V\psi_k^0(\cdot, \cdot, \alpha\omega')]}(x', y, \alpha\omega')\widetilde{u_k}(\alpha\omega')d\omega' dx' dy.$$

Remark: It should be mentioned the matrix-form of the scattering amplitude which differs radically from the classical one.

4 Asymptotic Behaviour of the Generalized Eigenfunctions

The functions $w_j^\pm$ see (2.2) are defined by

$$\begin{aligned} w_j^\pm(x', y, p) &= -R_0^\pm(\lambda_j(|p|))(V\psi_j^0)(x', y, p) + R_0^\pm(\lambda_j(|p|))VR^\pm(\lambda_j(|p|))(V\psi_j^0)(x', y, p) \\ &= I_{1,j}(x', y, p) + I_{2,j}(x', y, p). \end{aligned}$$

We are looking for the asymptotic behaviour of each term of this sum. Let us consider the k^{th} component of the first term

$$[I_{1,j}(x',y,p)]_k = \int_{\mathbb{R}^n} \psi_k^0(x',y,p) \frac{1}{|p'|^2+k^2-\lambda_j-i0} [V\widetilde{\psi_j^0(\cdot,p)}](p')dp'.$$

Then

$$[I_{1,j}(x',y,p)]_k = (2\pi)^{-3n}(\frac{2}{\pi})^{3/2} sin(ky) \int_{\mathbb{R}^n} \frac{e^{ix'\cdot p'}}{|p'|^2+k^2-\lambda_j-i0} f_1(p')dp'$$

$$\text{with} \quad f_1(p') = \int_{\mathbb{R}^n\times(0,\pi)} e^{i\widetilde{x'}\cdot(p-p')} \sin(k\widetilde{y}) \sin(j\widetilde{y}) V(\widetilde{x'},\widetilde{y}) d\widetilde{x'}d\widetilde{y}.$$

Remark: To obtain an asymptotic development of $[I_{1,j}(x',y,p]_k$, we use the lemma A1 of [2]. For this we have to replace $f_1 \in C^\infty(\mathbb{R}^n)$ by a $C_0^\infty(\mathbb{R}^n)$ function.
Let $p_0' = \alpha\omega'$, (p_0' verify $|p_0'|^2+k^2 = \lambda_j(|p|)$), we define the $C_0^\infty(\mathbb{R}^n)$ function χ equal to 1 in a neighborough of p_0' and 0 outside. Then

$$[I_{1,j}(x',y,p)]_k = (2\pi)^{-2n}(\frac{2}{\pi})^{3/2} sin(ky)(J_1(x')+K_1(x'))$$

$$J_1(x') = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \frac{e^{ix'\cdot p'}\chi(p')}{|p'|^2+k^2-\lambda_j-i0} f_1(p')dp',$$

$$K_1(x') = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \frac{e^{ix'\cdot p'}(1-\chi(p'))}{|p'|^2+k^2-\lambda_j-i0} f_1(p')dp'.$$

Then, we apply the lemma A1 of [2] and we obtain

$$J_1(x') = C_{p_0'} \frac{e^{i|x'|\cdot|p_0'|}}{|x'|^{\frac{n-1}{2}}}(f_1(p_0') + \mathcal{O}(|x'|^{-1}))$$

where $C_{p_0'} = (4\pi)^{-1}\alpha^{\frac{n-3}{4}}(2\pi)^{\frac{3-n}{2}} e^{-i\pi\frac{n-3}{4}}$. Moreover, we prove that $K_1(x') = \mathcal{O}(|x'|^{-\frac{n+1}{2}})$. So the asymptotic behaviour of the first term of w_j^+, denoted $\overline{\overline{[I_{1,j}(x',y,p)]_k}}$, is given by

$$\begin{aligned} \overline{\overline{[I_{1,j}(x',y,p)]_k}} &= (\tfrac{2}{\pi})^{\frac{1+5n}{2}} \sin(ky)\alpha^{\frac{n-3}{2}} e^{-i\pi\frac{n-3}{4}} \frac{e^{i|x'|\cdot|p_0'|}}{|x'|^{\frac{n-1}{2}}} \\ &\times \int_{\mathbb{R}^n\times(0,\pi)} e^{i\widetilde{x'}\cdot(p-p_0')} \sin(k\widetilde{y})\sin(j\widetilde{y})V(\widetilde{x'},\widetilde{y})d\widetilde{x'}d\widetilde{y}. \end{aligned}$$

In the same way, we prove that the asymptotic behaviour of the second term of w_j^+, denoted $\overline{\overline{[I_{2,j}(x',y,p)]_k}}$, is given by

$$\begin{aligned} \overline{\overline{[I_{2,j}(x',y,p)]_k}} &= \frac{(2\pi)^{\frac{1-3n}{2}}}{\pi} \sin(ky)\alpha^{n-3}V(x',y)e^{-i\pi\frac{n-3}{4}} \frac{e^{i|x'|\cdot|p_0'|}}{|x'|^{\frac{n-1}{2}}} \\ &\times \int_{\mathbb{R}^n\times(0,\pi)} e^{-i\widetilde{x'}\cdot p_0'} \sin(k\widetilde{y}) R(\lambda_j(|p|)(V(\cdot,\cdot)\psi_j^0(\cdot,\cdot,p))(\widetilde{x'},\widetilde{y},p)d\widetilde{x'}d\widetilde{y}. \end{aligned}$$

In conclusion, we obtain the asymptotic behaviour of the k^{th} component of w_j^+:

$$\overline{\overline{[w_j^+(x',y,p)]_k}} = \overline{\overline{[I_{1,j}(x',y,p)]_k}} + \overline{\overline{[I_{2,j}(x',y,p)]_k}}. \tag{4.1}$$

So we have to link the far field pattern with the scattering amplitude. For this, we compare (3.5) with (4.1).

Proposition 4.1 *If we set*

$$[G(x', y, p)]_k = \sqrt{\frac{2}{\pi}}(2\pi)^{\frac{-n-1}{2}} \sin(ky)\alpha^{\frac{-n+1}{2}} e^{-i\pi\frac{n-1}{2}} \frac{e^{i|x'|\cdot\alpha}}{|x'|^{\frac{n-1}{2}}},$$

with $\alpha = (\lambda_j(|p|) - k^2)^{1/2}$*, we can write*

$$\psi_j^+(x', y, p) = \psi_j^0(x', y, p) + \sum_{k=1}^{N(\lambda)} \mathcal{A}_{j,k}(\omega, \omega', \lambda_j(|p|))[G(x', y, p)]_k + o(|x'|^{-\frac{n+1}{2}}).$$

This relation allows us to give a physical sense to the scattering amplitude.
Our goal is to prove that exists an injection between the scattering amplitude and the potential $V(X)$. For the layer $(\mathbb{R}^2 \times (0, \pi))$ [3] prove an uniqueness result but using the Dirichlet to Neumann map. In a strip $(\mathbb{R} \times (0, \pi))$ does not exist any result. We think that the knowledge of the scattering amplitude at fixed energy is not sufficient to determine the potential. We want to prove that a countable infinity of energies will be sufficient.

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